CONTROL OF AN OBJECT WITH DELAY

(OB UPRAVLENII OB'EKTOM S POSLEDEISTVIEM)

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We shall consider the problem of decay of a linear system with delay

$$\frac{dx(t)}{dt} = Ax(t) + Gx(t-\tau) + bu \tag{1}$$

which must, by means of the control $u = u(t)$, be transferred from the given initial state $x(t) = x^\circ(t)$ $(-\tau \leqslant t \leqslant 0)$ to the equilibrium state $x(t) \equiv 0$ $(T-\tau \leqslant t \leqslant T)$.

We shall limit ourselves to the simplest case when x is a two-dimensional vector, u is a scalar, A and G are constant matrices, b is a constant vector and $T = 3\tau$ (τ = const). In this case our problem has an elementary solution.

The most interesting situation is obtained, when the matrix G is nonsingular and vector b is not its characteristic vector, i.e. when the condition of the generality of position [1] is fulfilled. We shall investigate this case. Let vector c be a solution of the equation $Gc = b$. Vectors c and b are, by definition, not colinear. Consequently, they can be regarded as base vectors on the plane $\{x_1, x_2\}$ and we have $c = \{1,0\}$, $b = \{0,1\}$. In these coordinates matrix G has the form

$$G = \begin{pmatrix} 0 & g_{12} \\ 1 & g_{22} \end{pmatrix}, \qquad g_{12} \neq 0 \tag{1.1}$$

Function $u(t)$ will be a solution to our problem if and only if

$$bu(t) + Gx(t-\tau) = 0 \qquad (T-\tau \leqslant t \leqslant T) \tag{2}$$

i.e.

$$x_2(t) = 0 \qquad (T-2\tau \leqslant t \leqslant T-\tau) \tag{3}$$

$$\frac{dx_2(t)}{dt} = a_{21}x_1(t) + x_1(t-\tau) + g_{22}x_2(t-\tau) + u(t) = 0 \quad (T-2\tau \leqslant t \leqslant T-\tau) \tag{4}$$

$$\frac{dx_1(t)}{dt} = a_{11}x_1(t) + g_{12}x_2(t-\tau), \qquad x_1(T-\tau) = 0 \qquad (T-2\tau \leqslant t \leqslant T-\tau) \tag{5}$$

Conditions (4) and (2) together define $u(t)$ completely when $t \geqslant \tau$, condition (3) however must also be fulfilled, which means that (2) in this case, is, $u(t) = -x_1(t-\tau)$ $(T-\tau \leqslant t \leqslant T)$. In order to find $u(t)$ from (3) and (5) when $0 \leqslant t < \tau$, we make use of

$$\int_0^\tau x_{22}(\tau, \vartheta)\, u(\vartheta)\, d\vartheta = \gamma_1$$
$$e^{a_{11}\tau}\int_0^\tau x_{12}(\tau, \vartheta)\, u(\vartheta)\, d\vartheta + \int_0^\tau g_{12} e^{a_{11}(\tau-\zeta)}\left[\int_0^\zeta x_{22}(\zeta, \vartheta)\, u(\vartheta)\, d\vartheta\right] d\zeta = \gamma_2 \qquad (6)$$

where $x_{ij}(t, t_0)$ are the elements of the fundamental matrix $X[t, t_0]$ of the system $dx/dt = Ax$, while γ_1 and γ_2 are found from the initial function $x^\circ(t)$.

Transformation of the second equation of (6) yields

$$\int_0^\tau h_i(\vartheta)\, u(\vartheta)\, d\vartheta = \gamma_i \qquad (i = 1, 2)$$
$$h_1(\vartheta) = x_{22}(\tau, \vartheta), \qquad h_2(\vartheta) = g_{12}\int_\vartheta^\tau e^{a_{11}(\tau-\zeta)} x_{22}(\zeta, \vartheta)\, d\zeta + e^{a_{11}\tau} x_{12}(\tau, \vartheta) \qquad (7)$$

Equations (7) will have a solution $u(\vartheta)$ for any γ_1 and γ_2 if and only if the function $h_1(\vartheta)$ and $h_2(\vartheta)$ are linearly independent [2]. Last condition is fulfilled whenever the condition of the generality of position is fulfilled for A and b, i.e. whenever the vector b is not a characteristic vector of the matrix A. Determination of $u(t)$ from (7) can be performed, using well known methods (see eg. [2]).

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